PROPAGATION OF TWO-DIMENSIONAL PLASTIC WAVES IN A THICK PLATE

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A study is made of the propagation and interaction of two-dimensional waves of high amplitude in a thick plate. A monotonically decreasing pressure is applied to the surface of the plate. Deformations are assumed to be large; the problem is formulated and solved in Lagrangian variables. An approximate method for constructing the fronts of the shock waves is proposed. The pressure and particle velocity at an arbitrary point and at an arbitrary instant of time are determined by the method of characteristics. A numerical example is given.

1. We consider an infinite plate with the free surfaces x = 0 and x = h (x is a Lagrangian coordinate of the medium). At the surface x = 0 we assume that a dynamic compressive force, varying according to the law $\sigma = \sigma_0 f(t)$, is acting; the symbol $f(t)$ denotes a monotonically decreasing function with $f(0) = 1$. The stress-deformation dependence is approximated with the aid of the formulas (Fig. 1)

$$\sigma = L\left(\frac{1}{1-e}\right)^k + M \quad \text{for} \quad \sigma > \sigma_A \quad \sigma = E_1 e \quad \text{for} \quad \sigma < \sigma_A \tag{1.1}$$

Here L, M, and k are characteristic constants for the given material, and the quantity E_1 is calculated from the formula

$$E_1 = \frac{1}{e_A}\left[L\left(\frac{1}{1-e_A}\right)^k + M\right] \tag{1.2}$$

The slope of the tangent at the point A is obtained from the formula

$$E_2 = \left(\frac{d\sigma}{de}\right)_{e=e_A} = \frac{kL}{(1-e_A)^{k+1}} \tag{1.3}$$

We assume that the pressure applied to the plate is sufficiently high so that loading and unloading of the material can be assumed to take place along a single path (hydrodynamic model of the medium). We neglect any effect of the temperature and also of the deformation rate on the mechanical parameters of the material.

2. The equations of motion in the Lagrangian coordinates x, t have the form (v is particle speed, ρ is material density, and ρ_0 is the density of the undeformed state)

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial \sigma}{\partial x} = 0, \quad \frac{\partial \rho}{\partial t} + \frac{\rho^2}{\rho_0}\frac{\partial v}{\partial x} = 0 \tag{2.1}$$

This system has the characteristics $dx = \pm a(e)dt$, where the Lagrangian wave speed $a(e)$ is given by the formulas

$$a(e) = \begin{cases} \sqrt{E_1/\rho_0} & \text{for} \quad \sigma < \sigma_A \\ \sqrt{kL/\rho_0}\,(1-e)^{-1/2(k+1)} & \text{for} \quad \sigma > \sigma_A \end{cases} \tag{2.2}$$

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If $\sigma > \sigma_A$, the conditions on the characteristics have the form

$$v + K(1-e)^{1/2(1-k)} = R_1 \quad \text{for} \quad dx = a(e)\,dt$$
$$v - K(1-e)^{1/2(1-k)} = R_2 \quad \text{for} \quad dx = -a(e)\,dt \qquad \left(K = \frac{2}{k-1}\left(\frac{kL}{\rho_0}\right)^{1/2}\right) \tag{2.3}$$

At the wave fronts the jump conditions are satisfied, i.e.,

$$[v] = a\,[e], \qquad [\sigma] = \rho_0 a\,[v] \tag{2.4}$$

For the case under consideration regions on the xt-plane appear, distributed as shown schematically in Fig. 2.

Two shock waves appear: one shock wave arises during loading of the material owing to the concavity of the portion AB in the diagram of Fig. 1; since $E_2 < E_1$, yet another shock wave arises during unloading of the material (upon passing through the point A of Fig. 1). The fronts of these shock waves are shown in Fig. 2 as the curves 0BC and AC. Calculations show that in many cases the quantities v and e change very little along the positive characteristics. An approximate method for constructing the shock front may be based on this fact whereby the quantities v and e are assumed to be constant along the positive characteristics. We remark that such a method was in essence applied in [1] by Lyakhov and Polyakova. In another paper we shall present a more precise method for constructing the shock front, one in which an estimate is given of the precision attained by the approximate method.

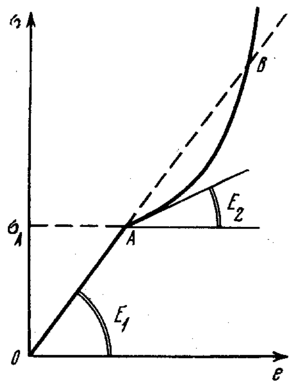

Fig. 1

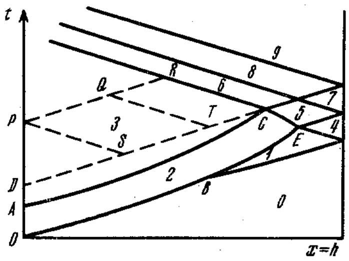

Fig. 2

We now determine the shock front according to the following scheme: If the quantities v and e are constant along the positive characteristics, then these characteristics are straight lines. In the case considered here, the function e = e(t) is given on the axis x = 0. We choose a sufficiently small time interval Δt and plot on the axis x = 0 the points $M_n = n\Delta t$, where n = 1, 2, 3, . . . (Fig. 3); through these points we draw the characteristics $x = a(e)t + C$ in the positive direction. If the shock wave passes through the point M and its initial speed a_{*0} is known, we can construct the initial portion MD_1 of the shock wave according to the formula $x = a_{*0}(t - t_M)$.

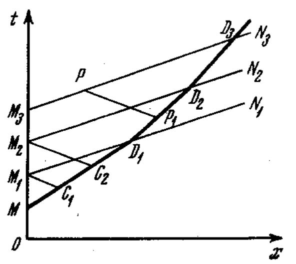

Fig. 3

At the point D_1 the jump conditions (2.4) have the form

$$\sigma_{D_1}{}^+ - \sigma_{D_1}{}^- = \rho_0 a_{*1}{}^2 (e_{D_1}{}^+ - e_{D_1}{}^-), \quad v_{D_1}{}^+ - v_D{}^- = a_{*1}(e_{D_1}{}^+ - e_{D_1}{}^-) \tag{2.5}$$

The plus and minus superscripts refer to values of quantities after and before passage of the shock front, respectively, and a_{*1} is the shock wave speed at the point D_1.

Since we regard the values $e_{D_1}{}^-$, $v_{D_1}{}^-$, $\sigma_{D_1}{}^-$ as known, and since, in addition, we have

$$v_{D_1}{}^+ = v_{M_1}, \quad e_{D_1}{}^+ = e_{M_1}$$

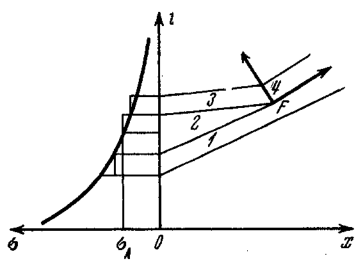

Fig. 4

we can then determine the quantities v_{M_1} and a_{*1} from the system (2.6). With slope a_{*1} we draw the next portion of the shock wave up to its intersection with the characteristic M_2N_2 at the point D_2 and we repeat the computations indicated above. All portions of the shock curve may be constructed in this way. We remark that with the present method the conditions (2.3) are not satisfied on the negative characteristics. This latter circumstance makes it possible to verify the precision attained with the approximate solution. To do this, we construct several negative characteristics M_1C_1, M_2C_2, . . . , then find the values of v and e at the points of intersection with the positive characteristics, and then check to see how much they deviate from the conditions (2.3).

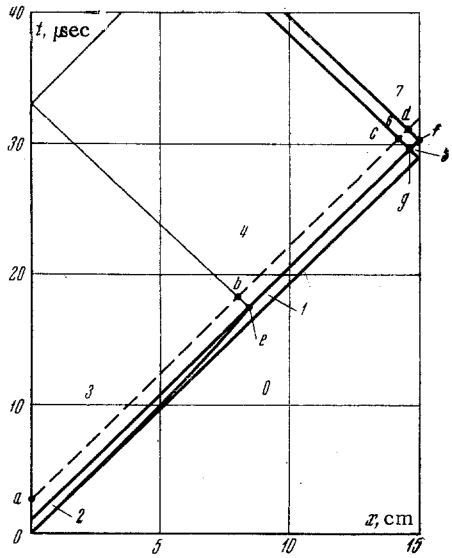

Fig. 5

To apply the method of constructing the shock front outlined above, the slope a_{*0} of the initial portion of the wave must be known. For a loading wave this quantity may be readily found; as is evident from Fig. 2, near the point O $\sigma^- = v^- = 0$, $\sigma^+ = \sigma_0$, and in accord with the jump conditions (2.4) we have

$$a_{*0} = \sqrt{\sigma_0 / \rho_0 e_0} \tag{2.6}$$

Determination of the quantity a_{*0} for an unloading wave is more involved, since here the quantities σ^- and v^- are not known. To solve this problem we recommend the following scheme:

We choose a small quantity $\Delta\sigma$ and replace the curve $\sigma = \sigma(t)$ of the diagram close to the point $\sigma = \sigma_A$ by a step-curve (Fig. 4), the height of each step being $\Delta\sigma$. In the xt plane there now appear regions 1, 2, 3 in which the parameters σ and v have constant values (Fig. 4), where $\sigma_1 = \sigma_A + \Delta\sigma$, $\sigma_2 = \sigma_A$, $\sigma_3 = \sigma_A - \Delta\sigma$.

The front 1-2 has the slope $a = a|_{\sigma = \sigma_A + \Delta\sigma}$, and the front 2-3 has the slope $a_1 = \sqrt{E_1/\rho_0}$. Since $a_1 > a$, both waves meet at the point F. This collision gives rise to a refracted wave 1-4 and a reflected wave 3-4. The parameters σ and v are constant also in the region 4. Apparently, $\sigma_4 < \sigma_A$, since otherwise the shock front 1-4 would not have arisen. Since $\sigma_3 < \sigma_A$, the front between the regions 3-4 propagates at the speed a_1.

We now deduce formulas for determining the particle speed in the regions 1-4, the stress σ_4, and the speed of the shock front 1-4. To do this we write out the jump conditions at the wave fronts (a_{*0} is the speed of the shock wave 1-4)

$$\begin{aligned} v_2 - v_1 &= a\,(e_2 - e_1) \\ v_3 - v_2 &= a_1\,(e_3 - e_2) \\ v_4 - v_3 &= -\,a_1\,(e_4 - e_3) \\ v_4 - v_1 &= a_{*0}\,(e_4 - e_1) \\ \sigma_4 - \sigma_1 &= \rho_0 a_{*0}^2\,(e_4 - e_1) \end{aligned} \tag{2.7}$$

On the basis of the relations (1.1) and (1.3) we have

$$\begin{gathered} e_1 \approx e_A + \left(\frac{de}{d\sigma}\right)_{\sigma=\sigma_A} \Delta\sigma = e_A + \frac{\Delta\sigma}{E_2} = \frac{\sigma_A}{E_1} + \frac{\Delta\sigma}{E_2} \\ e_2 = e_A = \frac{\sigma_A}{E_1}, \quad e_3 = \frac{\sigma_A - \Delta\sigma}{E_1}, \quad e_4 = \frac{\sigma_4}{E_1} \end{gathered} \tag{2.8}$$

We also introduce the notation

$$\varkappa = \frac{\sigma_A - \sigma_4}{\Delta\sigma} \tag{2.9}$$

The system (2.7) now acquires the form

$$\begin{gathered} v_2 - v_1 = -\,a\frac{\Delta\sigma}{E_2}, \quad v_3 - v_2 = -\,a_1\frac{\Delta\sigma}{E_1} \\ v_4 - v_3 = -\,a_1\frac{1-\varkappa}{E_1}\Delta\sigma, \quad v_4 - v_1 = -\,a_{*0}\left(\frac{1}{E_2} + \frac{\varkappa}{E_1}\right)\Delta\sigma \end{gathered} \tag{2.10}$$

$$\rho_0 a_{*0}^2 = \frac{E_1 E_2 (1 + \varkappa)}{E_1 + E_2 \varkappa} \tag{2.11}$$

For a sufficiently small step $\Delta\sigma$ we can assume that $a \approx \sqrt{E_2/\rho_0}$; if now we add the first three of Eqs. (2.10) and subtract the last one, we obtain

$$\frac{1}{\sqrt{E_2}} + \frac{2-\varkappa}{\sqrt{E_1}} - a_{*0}\sqrt{\rho_0}\left(\frac{1}{E_2} + \frac{\varkappa}{E_1}\right) = 0 \tag{2.12}$$

From Eqs. (2.11), (2.12) we determine the quantities $\varkappa$ and a_{*0}. If, in addition, we introduce for conciseness the notation $\gamma = E_2/E_1$, we find that

$$\varkappa = \frac{4(\gamma + \sqrt{\gamma})}{1 + 2\sqrt{\gamma} + 5\gamma} \tag{2.13}$$

$$a_{*0}\sqrt{\rho_0} = \frac{\sqrt{E_2}(1 + 3\sqrt{\gamma})}{1 + \sqrt{\gamma} + 2\gamma} \tag{2.14}$$

Formula (2.14) determines the slope of the initial portion of the shock wave which arises with unloading of the material.

We now proceed to a determination of the parameters σ and v in the regions of Fig. 2. In those portions of the shock wave of loading, where $\sigma^+ > \sigma_B$ (the portion OB of Fig. 2; σ_B is the stress at the point B of Fig. 1), the passage into the region 2 proceeds directly from the quiescent region 0. If, however, $\sigma^+ < \sigma_B$, then between the regions 0 and 2 there must, in addition, be a region 1 where the parameters have constant values, namely,

$$\sigma_1 = \sigma_A, \quad v_1 = \frac{\sigma_A}{\rho_0 a_1} \tag{2.15}$$

The straight line 0-1 is a strong discontinuity curve, its inclination to the t axis being $a_1 = \sqrt{E_1/\rho_0}$. The front 0-2 is a shock front. At point B of Fig. 2, where $\sigma = \sigma_B$, a branching of the fronts takes place. If initially the stress σ is less than σ_B (i.e., $\sigma_0 < \sigma_B$), the point B coincides with point O and the branching of the fronts occurs at the origin. In region 2, where $\sigma > \sigma_A$, the characteristics are not straight lines, thereby complicating the determination of the parameters in regions 2 and 3. The calculations are simplified significantly if the assumption is made that in regions 2 and 3 the quantities σ and v are constant along the positive characteristics.

The wave 0-1 reflects from the wall x = h and a region 4 appears, where

$$\sigma_4 = 0, \quad v_4 = \frac{2\sigma_A}{\rho_0 a_1} \tag{2.16}$$

As a result of the interaction of the waves 1-4 and 1-2, two new waves appear at the point E, where the refracted wave 4-5 is a curve of strong discontinuity and the reflected wave 2-5 is a shock wave. In subsequent interactions there appear the additional regions 6-9. Only in region 2 do we have $\sigma_2 > \sigma_A$, in all the remaining regions the stress is less than σ_A.

Constancy of the quantities σ and v is assumed only for the region 2. In the regions 3, 5-9 values of these quantities are easily determined by the method of characteristics. As for region 3, we initially determine the values of v in the triangular region ACD and then subsequently for the remaining part of region 3.

To illustrate the method we determine the quantities e and v for the points P, Q, and R of Fig. 2, assuming, moreover, that the shock wave AC has already been constructed and that the values of e and v have also been found in the region ACD. Through the points P and Q we draw the positive and negative characteristics; they are straight lines with slopes $\pm a_1 = \pm\sqrt{E_1/\rho_0}$. For these characteristics the following conditions must be satisfied:

$$\begin{aligned} v_Q + a_1 e_Q &= v_P + a_1 e_P \\ v_Q - a_1 e_Q &= v_T - a_1 e_T \\ v_P - a_1 e_P &= v_S - a_1 e_S \end{aligned} \tag{2.17}$$

Since the parameters in the region ACD have already been determined, the quantities v_T, e_T, v_S, and e_S are known. In addition, e_P is known from the loading schedule $\sigma = \sigma(t)$ on the surface x = 0. Consequently, from the system (2.17) we may determine the unknown quantities v_P, v_Q, and e_Q. We now proceed to the point R. We shall interpret the jump curve 4-7 as a double characteristic. The conditions for the characteristics now have the form

TABLE 1

x	$t\cdot10^{-6}$	$a_*\cdot10^{-6}$	$\sigma^-\cdot10^{-5}$	$\sigma^+\cdot10^{-5}$
0	0.743	0.4708	1.650	1.650
0.309	1.398	0.4870	1.708	1.527
0.510	1.811	0.4850	1.749	1.469
0.866	2.546	0.4845	1.806	1.368
1.300	3.441	0.4890	1.851	1.253
1.614	4.084	0.4901	1.881	1.181
2.017	4.909	0.4914	1.911	1.096
2.489	5.869	0.4926	1.942	1.007
3.085	7.079	0.4939	1.974	0.907
3.816	8.558	0.4950	2.006	0.801
4.859	10.665	0.4963	2.038	0.673
6.246	13.460	0.4975	2.072	0.538
8.179	17.345	0.4986	2.105	0.396

TABLE 2

x	$t\cdot10^{-6}$	$a_*\cdot10^{-6}$	$\sigma^+\cdot10^{-5}$
0	0	0.5004	3
0.572	1.150	0.4904	2.768
0.924	1.867	0.4855	2.658
1.427	2.903	0.4808	2.554
1.784	3.638	0.4780	2.493
2.286	4.687	0.4752	2.433
3.018	6.226	0.4716	2.356
3.622	7.506	0.4694	2.309
4.322	8.998	0.4672	2.263
4.736	9.885	0.4651	2.218
6.237	13.113	0.4630	2.174
7.225	15.226	0.4612	2.139
8.677	18.334	0.4595	2.105

TABLE 3

Points	$\sigma\cdot10^{-5}$		$v\cdot10^{-5}$	
	−	+	−	+
a	0.3326		0.0525	
b	0.3266	0.4131	0.0540	0.0315
c	0.4131	−1.1744	0.0315	0.4459
d	−1.1744	−0.0087	0.4459	0.1416
e	0.3771	0.4217	0.0672	0.0337
g	0.4217	−1.1657	0 0337	0.4481
f	−1.1657	0	0.4481	0.1438

$$v_R^- + a_1 e_R^- = v_Q + a_1 e_Q$$
$$v_R^- - a_1 e_R^- = v_C^- - a_1 e_C^-$$
$$v_R^+ - a_1 e_R^+ = v_C^+ - a_1 e_C^+ \tag{2.18}$$

Moreover, it will also be necessary to satisfy the jump condition

$$v_R^+ - v_R^- = a_1 (e_R^+ - e_R^-) . \tag{2.19}$$

From Eqs. (2.18), (2.19) we determine the quantities v_R^-, v_R^+, e_R^-, e_R^+. If the deformations e_Q, e_R^-, e_R^+ are already known, the corresponding stress can be determined on the basis of the $\sigma - e$ diagram.

In this way the values of σ, e, and v can be found for an arbitrary point of the diagram of Fig. 2.

We remark that the distribution of regions shown in Fig. 2 is not the only one possible. If the plate is sufficiently thick, the shock front of loading may intersect the shock front of unloading even before the boundary x = h is reached. We take up the consideration of this case in an example, which we solve in Section 3.

3. Consider a plate of thickness h = 15 cm. The stress applied to its surface x = 0 varies according to the law

$$\sigma = 300\,000 \exp(-0.805\cdot10^6\, t)$$

For the values of the material parameters L, M, k, and σ_A we take

$$L = 142.5\cdot10^3,\ M = -67.1\cdot10^3,\ \sigma_A = 165\,000 \text{ bar},\ k = 6$$

Before proceeding to a solution of the problem we determine the quantity σ_B (cf. Fig. 1). Upon making the computations we find that $\sigma_B = 3.28 \cdot 10^5$ bar. Since in this case $\sigma < \sigma_B$, the region 1 of Fig. 2 is already present at the initial instant t = 0. Next we construct the fronts of the shock waves 1-2 and 2-3 (Fig. 5), the initial slope of the front 2-3 being calculated from formula (2.14): $a_{*0} = 0.4708 \cdot 10^6$ cm/sec. The computational results are presented in Tables 1 and 2, the data in Table 1 corresponding to the front

1-2 and that in Table 2 to the front 2-3. In these tables x and t are coordinates of a point of the shock front, a_* is the wave speed at the point, $\sigma^{\mp}$ are the stresses on the negative and positive sides of the front (for the front 1-2 we have $\sigma^- = 1.65 \cdot 10^5$ bar.

The disposition of the fronts and the regions in the xt plane are shown in Fig. 5. As is evident from Fig. 5, the region 2, where $\sigma > \sigma_A$, is very small; in all the remaining regions we have $\sigma < \sigma_A$ and, consequently, the characteristics have the constant slopes $\pm a_1 = \pm 0.5516 \cdot 10^6$ [illegible]/sec. In order to study the variation of the parameters in the various regions, we also carried through calculations for the points a, b, c, d, e, f, g shown in Fig. 5; the data for these points is given in Table 3. The points e-f are regarded as lying on the positive side of the fronts 1-4 and 5-6; the plus and minus signs in Table 3 indicate on which side of the fronts 3-4, 4-6, or 6-7 the corresponding point lies.

One can draw the following conclusions from the data of Table 3:

1) On the intervals b-c, c-d, e-g, g-f of a positive characteristic the parameters σ and v show negligible variation; this situation holds only for the initial portion a-b.

2) The phenomenon of spall takes place at point g, the thickness of the spall that flies off being about 0.5 cm.

3) If the inequality $x_e < h$ is satisfied (x_e is the coordinate of point e), then when spall occurs the spall thickness is independent of the general plate thickness h.

LITERATURE CITED

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